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LETTER TO THE EDITOR

Invariants of the quantum supergroup $U_q(gl(m/1))$

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Abstract. A spectral parameter dependent solution of the graded Yang–Baxter equation is obtained, which in $U_q(gl(m/1))\otimes End(V)$ with V being the vector module of $U_q(gl(m/1))$. Invariants of this quantum supergroup are constructed using this solution and a general method developed in an earlier publication.

Quantum supergroups [1, 2] are the supersymmetric generalizations of quantum groups [3, 4], which are obtained by quantizing the Lie bi-super algebraic structures [5, 6] of the basic classical Lie superalgebras. They constitute a class of $\mathbb{Z}_2$-graded quasitriangular Hopf algebras, which are interesting from a purely mathematical point of view. It is also important physically to explore the structure of quantum supergroups because of their close connection with the graded Yang–Baxter equation.

In an earlier publication [7] we presented a general method for constructing invariants of quantum supergroups. As demonstrated by the $U_q(gl(2/1))$ example, the method enables one to construct quantum supergroup invariants in a systematical way, provided that $(id\otimes\pi)R$ and $(id\otimes\pi)R^T$ are known explicitly for some nontrivial representation π, where R is the universal R-matrix, and R^T its transpose.

As a first step towards the construction of the invariants of the quantum supergroup $U_q(gl(m/n))$, we study in this letter the special case $U_q(gl(m/1))$ using the method developed in [7]. The main difficulty in doing this is that the universal R-matrix of this quantum supergroup is only formally known, and it is a highly nontrivial and laborious exercise to obtain an explicit formula for it. Fortunately, for the vector representation π furnished by the vector module V, $(\pi\otimes id)R$ and $(\pi\otimes id)R^T$ of $U_q(gl(m/1))$ can be constructed by first building a solution $L(x)\in U_q(gl(m/1))\otimes End(V)$ of the graded Yang–Baxter equation, and then examining its limits when the spectral parameter x goes to ∞ and 0. This rather indirect approach to the construction of $(\pi\otimes id)R$ and $(\pi\otimes id)R^T$ avoids the very difficult problem of explicitly evaluating the universal R-matrix, which, though important, is beyond the scope of this letter. Also, the $L(x)$-operator obtained is interesting in its own right: for each irreducible representation of $U_q(gl(m/1))$ it yields an integrable lattice model.

Using the L-operator, the invariants of $U_q(gl(m/1))$ are constructed following the procedure given in [7]. For the sake of concreteness we work out the explicit expressions of some of the invariants and their eigenvalues in arbitrary irreducible highest weight representations.

The quantum supergroup $U_q(gl(m/1))$ is the one-parameter deformation of the universal enveloping algebra of the Lie superalgebra $gl(m/1)$. It is a unital algebra generated by $E^a_{a\pm1}$, E^a_a, a, $a\pm1=1,2,\ldots,m+1$, subject to the constraints

$$
\begin{aligned}
&[E^a_a, E^b_b]=0\\
&[E^a_a, E^b_{b\pm1}]=(\delta_{ab}-\delta_{ab\pm1})E^b_{b\pm1}\\
&[E^a_{a+1}, E^{b+1}_b\}=\delta_{ab}(q^{h_a}-q^{-h_a})/(q-q^{-1})\\
&(E^m_{m+1})^2=(E^{m+1}_m)^2=0 \qquad\qquad (1)\\
&E^a_{a\pm1}E^b_{b\pm1}=E^b_{b\pm1}E^a_{a\pm1} \qquad |a-b|\geqslant 2\\
&(E^a_{a+1})^2E^{a\pm1}_{a\pm1+1}-(q+q^{-1})E^a_{a+1}E^{a\pm1}_{a\pm1+1}E^a_{a+1}+E^{a\pm1}_{a\pm1+1}(E^a_{a+1})^2=0 \qquad a\neq m\\
&(E^{a+1}_a)^2E^{a\pm1+1}_{a\pm1}-(q+q^{-1})E^{a+1}_aE^{a\pm1+1}_{a\pm1}E^{a+1}_a+(e^{a+1}_a)^2E^{a\pm1+1}_a=0 \qquad a\neq m
\end{aligned}
$$

where

$$
\begin{aligned}
&[E^a_{a+1}, E^{b+1}_b\}=E^a_{a+1}E^{b+1}_b-(-1)^{[a+1][b+1]}E^{b+1}_bE^a_{a+1}\\
&h_a=E^a_a-(-1)^{[a+1]}E^{a+1}_{a+1}.
\end{aligned}
$$

In (1)

$$
[a]=\begin{cases}0 & a\neq m+1\\ 1 & a=m+1\end{cases}
$$

and we will say that the elements E^m_{m+1} and E^{m+1}_m are odd, while the rest appearing in (1) are even.

The algebra $U_q(gl(m/1))$ admits a co-multiplication Δ: $U_q(gl(m/1))\to U_q(gl(m/1))\otimes U_q(gl(m/1))$, defined for the elements $E^a_{a\pm1}$, E^a_a by

$$
\begin{aligned}
&\Delta(E^a_{a+1})=E^a_{a+1}\otimes q^{h_a}+1\otimes E^a_{a+1}\\
&\Delta(E^{a+1}_a)=E^{a+1}_a\otimes 1+q^{-h_a}\otimes E^{a+1}_a \qquad\qquad (2)\\
&\Delta(E^a_a)=E^a_a\otimes 1+1\otimes E^a_a
\end{aligned}
$$

a co-unit ε: $U_q(gl(m/1))\to\mathbb{C}$ such that

$$
\varepsilon(E^a_{a\pm1})=\varepsilon(E^a_a)=0 \qquad \varepsilon(1)=1
$$

and an anti-pode S: $U_q(gl(m/1))\to U_q(gl(m/1))$, which is an algebraic anti-homomorphism defined by

$$
\begin{aligned}
&S(E^a_{a+1})=-E^a_{a+1}q^{-h_a}\\
&S(E^{a+1}_a)=-q^{h_a}E^{a+1}_a \qquad\qquad (3)\\
&S(E^a_a)=-E^a_a.
\end{aligned}
$$

For later use we define the elements E^a_b, $\bar{E}^a_b$, a, $b=1,2,\ldots,m+1$, recursively by the following relations

$$
\begin{aligned}
&E^a_b=E^a_cE^c_b-q^{-1}E^c_bE^a_c\\
&E^b_a=E^b_cE^c_a-qE^c_aE^b_c \qquad a<c<b. \qquad\qquad (4)\\
&\bar{E}^a_b=\bar{E}^a_c\bar{E}^c_b-q\bar{E}^c_b\bar{E}^a_c\\
&\bar{E}^b_a=\bar{E}^b_c\bar{E}^c_a-q^{-1}\bar{E}^c_a\bar{E}^b_c
\end{aligned}
$$

It is easy to convince oneself that the above definitions are self-consistent.

The vector representation π of $U_q(gl(m/1))$ is not deformed in the following sense

$$\pi(E^a_{a\pm1}) = e^a_{a\pm1} \qquad \pi(E^a_a) = e^a_a \tag{5}$$

where e^a_b are the matrices

$$(e^a_b)_{\alpha\beta} = \delta_{a\alpha}\delta_{b\beta}.$$

Let us now construct the following equations for an invertible $L(x) \in U_q(gl(m/1)) \otimes \mathrm{End}(V)$,

$$L(x)(\mathrm{id}\otimes\pi)\Delta(\omega) = (\mathrm{id}\otimes\pi)\Delta'(\omega)L(x) \qquad \forall\omega\in U_q(gl(m/1)) \tag{6a}$$

$$L(x)\{x^{-2}E_0\otimes\pi(q^{-E^{m+1}_{m+1}-E^1_1}) + 1\otimes\pi(E_0)\} = \{x^{-2}E_0\otimes 1 + q^{-E^{m+1}_{m+1}-E^1_1}\otimes\pi(E_0)\}L(x) \tag{6b}$$

where V is the vector module of $U_q(gl(m/1))$, E_0 is defined by

$$E_0 = q^{-E^1_1+E^{m+1}_{m+1}}E^{m+1}_1$$

and $\Delta' = T\cdot\Delta$ represents the opposite co-multiplication with T: $U_q(gl(m/1))\otimes U_q(gl(m/1)) \to U_q(gl(m/1))\otimes U_q(gl(m/1))$ the twisting map such that for any two elements $u, v \in U_q(gl(m/1))$,

$$T(u\otimes v) = \begin{cases} -v\otimes u & \text{both } u \text{ and } v \text{ are odd} \\ v\otimes u & \text{otherwise.} \end{cases}$$

Equation (6) admits at most one solution up to scalar multiples, and a solution of it automatically satisfies the graded Yang-Baxter equation [2, 4]. By analysing the semi-classical limit of (6) we can see that for large $|x|$, $L(x)$ is proportional to $(\mathrm{id}\otimes\pi)R$, and when x is near zero, $L(x)$ is proportional to $(\mathrm{id}\otimes\pi)R^{-\mathrm{T}}$, with $R^{-\mathrm{T}} = T(R^{-1})$, where T is the twisting map defined above.

In order to solve (6) we first define

$$\begin{aligned} L^{(+)} &= q^{\sum_a E^a_a\otimes e^a_a(-1)^{[a]}}\left\{1\otimes I + (q-q^{-1})\sum_{a<b} E^a_b\otimes e^b_a(-1)^{[b]}\right\} \\ L^{(-)} &= \left\{1\otimes I - (q-q^{-1})\sum_{a<b} E^b_a\otimes e^a_b\right\}q^{-\sum_a E^a_a\otimes e_a(-1)^{[a]}} \end{aligned} \tag{7}$$

then show that:

Lemma.

$$L^{(\pm)}(\mathrm{id}\otimes\pi)\Delta(\omega) = (\mathrm{id}\otimes\pi)\Delta'(\omega)L^{(\pm)} \qquad \forall\omega\in U_q(gl(m/1)). \tag{8}$$

Proof. Let us consider $L^{(+)}$ first. Define

$$\Phi = 1\otimes I - (q-q^{-1})\sum_{i=1}^{m} E^i_{m+1}\otimes e^{m+1}_i$$

$$L^{(+)}_0 = q^{\sum_{i=1}^m E^i_i\otimes e^i_i}\left\{1\otimes I + (q-q^{-1})\sum_{1\le i<j\le m} E^i_j\otimes e^j_i\right\}.$$

Then $L^{(+)}$ can be expressed in terms of Φ and $L^{(+)}_0$ as

$$L^{(+)} = q^{-E^{m+1}_{m+1}\otimes e^{m+1}_{m+1}}L^{(+)}_0\Phi.$$

Note that $L^{(+)}_0$ is nothing else but $(\mathrm{id}\otimes\pi)R_0$, where R_0 is the universal R-matrix of the Hopf subalgebra $U_q(gl(m)) \subset U_q(gl(m/1))$. Also,

$$[\Phi, (\mathrm{id}\otimes\pi)\Delta(\omega)] = 0 \qquad \forall\omega\in U_q(gl(m))$$

thus equation (8) is satisfied by $L^{(+)}$ and all the elements of the Hopf subalgebra $U_q(gl(m))$. We therefore only need to show that E_m^{m+1} and E_{m+1}^m also satisfy (8) with $L^{(+)}$.

Straightforward manipulations give rise to the following relations

$$\Phi\{E_{m+1}^m \otimes q^{\pi(h_m)} + 1 \otimes e_{m+1}^m\}$$
$$= \left\{E_{m+1}^m \otimes q^{-\pi(h_m)} + 1 \otimes e_{m+1}^m - (q-q^{-1}) \sum_{i=1}^{m-1} E_{m+1}^i \otimes e_i^m\right\} \Phi$$

$$\sum_{1 \le i < j \le m} E_j^i \otimes e_i^j [E_{m+1}^m \otimes q^{-\pi(h_m)} + 1 \otimes e_{m+1}^m]$$
$$= [E_{m+1}^m \otimes q^{-\pi(h_m)} + 1 \otimes e_{m+1}^m] \sum_{1 \le i < j \le m} E_j^i \otimes e_i^j + \sum_{1 \le i < m} E_{m+1}^i \otimes e_i^m$$

which immediately lead to

$$L^{(+)}(\mathrm{id} \otimes \pi)\Delta(E_{m+1}^m) = (\mathrm{id} \otimes \pi)\Delta'(E_{m+1}^m) L^{(+)}.$$

The case for E_m^{m+1} can be proved in the same way. One can similarly demonstrate that (8) also holds for $L^{(-)}$.

A useful property of $L^{(-)}$ is

$$(L^{(-)})^{-1} = (S \otimes \mathrm{id}) L^{(-)}$$
$$= q^{\sum_a E_a^a \otimes e_a^a (-1)^{[a]}} \left\{1 \otimes I + (q-q^{-1}) \sum_{a<b} \bar{E}_a^b \otimes e_b^a\right\} \tag{9}$$

which can be easily proved by using the following relations

$$\bar{E}_a^b = E_a^b + (q-q^{-1}) \sum_{a<c<b} E_a^c \bar{E}_c^b \qquad a<b$$
$$S(E_a^b) = -q^{E_a^a - (-1)^{[b]} E_b^b} \bar{E}_a^b \qquad a<b$$

Now we come to the main result of this section:

Theorem 1. Let

$$L(x) = xL^{(+)} - x^{-1}L^{(-)}. \tag{10}$$

Then $L(x)$ satisfies equation (6), and

$$L^{(+)} = (\mathrm{id} \otimes \pi)R \qquad (L^{(-)})^{-1} = (\mathrm{id} \otimes \pi)R^{\mathrm{T}} \tag{11}$$

with R the universal R-matrix of $U_q(gl(m/1))$.

Proof. The second part of the theorem, i.e. equation (11), follows from the first part. In order to prove the first part, it suffices to demonstrate that $L(x)$ satisfies equation (6*b*) because of the lemma. This can be easily carried out by direct computations, thus we omit the details here.

We now construct invariants of $U_q(gl(m/1))$ using the results of the last section. Let us first introduce some new notation:

$$X_b^a = q^{(-1)^{[b]} E_b^b} E_b^a \qquad a<b$$
$$X_a^b = q^{(-1)^{[a]} E_a^a} \bar{E}_a^b \qquad a<b$$
$$X_a^a = 2(-1)^{[a]}(q^{(-1)^{[a]} E_a^a} - 1)/(q-q^{-1}).$$

Defining

$$\Gamma = (L^{(-1)})^{-1} L^{(+)}$$

we have

$$\Gamma = 1 \otimes I + (q - q^{-1}) \sum_{a,b} X_b^a \otimes e_a^b (-1)^{[b]}$$
$$+ (q - q^{-1})^2 \sum_{a \geq b,c} X_b^a X_a^c \otimes e_c^b (-1)^{([a]+[b])([b]+[c])} 2^{-\delta ab - \delta ac}. \tag{12}$$

Following the general result of [7], we have:

Theorem 2. Let

$$C_k = \mathrm{Str}_\pi \left\{ [1 \otimes q^{-\pi(2h\rho)}] \left(\frac{\Gamma - 1 \otimes I}{q - q^{-1}} \right)^k \right\} \qquad k \in \mathbb{Z}_+ \tag{13}$$

with

$$\pi(2h\rho) = \sum_{i=1}^{m} (m - 2i) e_i^i - m e_{m+1}^{m+1}$$

then C_k, $\forall k \in \mathbb{Z}_+$, belongs to the centre of $\mathrm{U}_q(\mathrm{gl}(m/1))$.

To explicitly compute the invariants, it is convenient to introduce an $(m+1)$-dimensional Minkowski space with basis vectors $\mathscr{E}_a$, $a = 1, 2, \ldots, m+1$, and a bilinear scalar product (,) such that

$$(\mathscr{E}_a, \mathscr{E}_b) = (-1)^{[b]} \delta_{ab}.$$

Note that this vector space is isomorphic to the dual space of the Cartan subalgebra of $\mathrm{gl}(m/1)$. Define

$$2\rho = \sum_{i=1}^{m} (m - 2i) \mathscr{E}_i + m \mathscr{E}_{m+1}.$$

Now it is a matter of straightforward manipulations to obtain the explicit forms of the invariants. For the first two, we have

$$C_1 = \sum_a X_a^a q^{(2\rho, \mathscr{E}_a)} + (q - q^{-1}) \sum_{a \geq b} X_b^a X_a^b (-1)^{[b]} q^{(2\rho, \mathscr{E}_b)} 4^{-\delta_{ab}} \tag{14}$$

$$C_2 = \sum_{a,b} X_b^a X_a^b q^{(2\rho, \mathscr{E}_b)} (-1)^{[b]}$$
$$+ (q - q^{-1}) \sum_{a \geq b,c} [X_b^a X_a^c X_c^b q^{(2\rho, \mathscr{E}_b)} + X_c^b X_b^a X_a^c q^{(2\rho, \mathscr{E}_c)} (-1)^{[b]+[c]}]$$
$$\times (-1)^{([a]+[b])([b]+[c])} 2^{-\delta_{ab} - \delta_{ac}}$$
$$+ (q - q^{-1})^2 \sum_{d,a \leq b,c} X_b^a X_a^c X_c^d X_d^b q^{(2\rho, \mathscr{E}_b)} (-1)^{([b]+[c])([d]+[c])+[b]}$$
$$\times 2^{-\delta_{ab} - \delta_{ac} - \delta_{db} - \delta_{dc}}. \tag{15}$$

The explicit forms of the higher-order invariants, although very easy to obtain, are rather complex, thus we will not spell them out here. However, it is worth pointing out that when q goes to 1, we have

$$C_k^{(0)} = \lim_{q \to 1} C_k = \sum_{\{c_i\}} X_{c_k}^{(0)c_1} X_{c_1}^{(0)c_2} \ldots X_{c_k-1}^{(0)c_k} (-1)^{\xi(c_i)} \tag{16}$$

$$\xi \equiv \sum_{i=1}^{k} \{[c_i] + ([c_i] + [c_k])([c_i] + [c_{i+1}])\} \qquad (\mathrm{mod}\ 2)$$

where the $X^{(0)}$'s are the standard gl$(m/1)$ elements in the tensor notation, or more precisely,

$$[X_b^{(0)a}, X_d^{(0)c}\} = \delta_{bc} X_d^{(0)a} - (-1)^{([a]+[b])([c]+[d])} \delta_{ad} X_b^{(0)c}.$$

The first $(m+1)$ $C^{(0)}$'s generate the entire centre of gl$(m/1)$. This is an indication that the C's also generate the centre of $U_q(\mathrm{gl}(m/1))$. We will return to this problem later.

Finally, it should be noted that in an arbitrary, irreducible highest weight representation with highest weight Λ, C_1 takes the following eigenvalue

$$\chi_\Lambda(C_1) = \sum_{a=1}^{m+1} (-1)^{[a]}(q^{2(\Lambda+\rho,\mathscr{E}_a)} - q^{2(\rho,\mathscr{E}_a)})/(q-q^{-1}). \tag{17}$$

We have constructed an $L(x) \in U_q(\mathrm{gl}(m/1)) \otimes \mathrm{End}(V)$ which satisfies the graded Yang–Baxter equation. When the spectral parameter goes to ∞ and 0, this L-operator yields $(\mathrm{id} \otimes \pi)R$ and $(\mathrm{id} \otimes \pi)R^{-\mathrm{T}}$, respectively, where R is the universal R-matrix of $U_q(\mathrm{gl}(m/1))$ and π is its vector representation. Using these objects, we have constructed invariants of this quantum supergroup following the general method developed in [7]. As we have mentioned earlier, we expect these invariants to generate the entire centre of $U_q(\mathrm{gl}(m/1))$. One indication of this is the fact that in the limit $q \to 1$, our invariants reduce to a complete set of generators of the centre of the Lie superalgebra gl$(m/1)$, as we have shown above. Also, there exists a similar construction for ordinary quantum group invariants using universal R-matrices and it can be shown in that case that this construction yields all the invariants of quantum groups. Thus we expect this to be true for quantum supergroups also.

The program of this paper can be extended to $U_q(\mathrm{gl}(m/n))$ for general m and n. Results are reported in a separate publication [8].

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References

[1] Kulish P and Reshetikhin N Yu 1989 *Lett. Math. Phys.* **18** 143
Chaichian M and Kulish P 1990 *Phys. Lett.* **234B** 72
Deguchi T, Fujii A and Ito K 1990 *Phys. Lett.* **236B** 242
Saleur H 1990 *Nucl. Phys.* B **336** 363

[2] Bracken A J, Gould M D and Zhang R B 1990 *Mod. Phys. Lett.* A **5** 331
Zhang R B, Gould M D and Bracken A J 1991 *Phys. Lett.* **257B** 133

[3] Drinfeld V G 1986 *Proc. Int. Congr. Math.* (Berkeley: MSRI) p 798
Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247

[4] Jimbo M 1986 *Commun. Math. Phys.* **102** 247

[5] Zhang R B, Gould M D and Bracken A J 1991 *J. Phys. A: Math. Gen.* **24** 1185

[6] Gould M D, Zhang R B and Bracken A J 1991 *Rev. Math. Phys.* **3** 223

[7] Zhang R B and Gould M D 1991 Universal R-matrices and invariants of quantum supergroups *J. Math. Phys.* in press

[8] Zhang R B Universal L-operator and invariants of the quantum supergroup $U_q(\mathrm{gl}(m/n))$ *Preprint* Australian National University